Decays and Scattering

- Decay Rates
- Cross Sections
- Calculating Decays
- Scattering
- Lifetime of Particles
Decay Rates

• There are THREE experimental probes of Elementary Particle Interactions
  - bound states
  - decay of particles
  - scattering of particles

• Decay of a particle
  - is of course a statistical process
  - represented by the average (mean) life time

• Remember:
  - elementary particles have no memory
  - probability of a given muon decaying in the next microsecond is independent of time of creation of muon (humans are VERY different !)

• Decay Rate $\Gamma$: probability per unit time that particle will disintegrate
Decay Rates

- Consider large collection of, for example, muons: $N(t)$

- Decay rate $dN = -\Gamma N dt$

  $$N(t) = N(0) \exp (-\Gamma t)$$

- Mean lifetime $\tau = 1/\Gamma$

- Most particles will decay by several routes
  - total decay rate $\Gamma_{tot} = \sum \Gamma_i$ ; $\tau = 1/\Gamma_{tot}$

- Branching ratio for $i$th decay mode
  - $BR(i) = \Gamma_i / \Gamma_{tot}$
Decay Width $\Gamma$

- Example: decay width of the $J/\psi$
- Shown is the original measurement at the SPEAR $e^+e^-$ Collider; the line width is larger ($\sim 3.4 \text{ MeV}$) due to energy spread in the beams
- Natural line width is 93 keV
- Lifetime is $\Delta E \Delta t \leq \hbar / 2\pi = 6.6 \times 10^{-16} \text{ eV}$
- $\Delta t \sim 7 \times 10^{-21} \text{ s}$
- We will see later – within a discussion of the quark model – why this lifetime is relatively long
Decay: Examples

- **Derive** $\tau = 1/\Gamma$
  - Hint: what fraction of original sample decays between $t$ and $t+dt$?
  - With this result: what is the initial probability $p(t) \, dt$ of any particle decaying between $t$ and $t+dt$?

- Mean lifetime $\tau = \int_{0}^{\infty} t(p) \, dt$

- **Starting with** $10^6$ muons at rest, how many will survive $10^{-5}$ seconds?
  - $\tau$ (muon) = 2.2 $10^{-6}$

- A muon with momentum $p=10\text{GeV}/c$ is produced in a cosmic ray collision with an oxygen nucleus of the atmosphere at 10km altitude. What is the probability of the muon reaching the earth’s surface?
Scattering: Cross sections

- What do we measure? What do we calculate?

- Classical analog:
  - aiming at target
  - what counts is the ‘size’ of the target

- Elementary particle scattering
  - size (‘cross-sectional area’) of particle is relevant
  - however: size is not ‘rigidly’ defined; size, area is ‘fuzzy’
  - effect of ‘collision’ depends on distance between projectile and scattering center
  - we can define an ‘effective cross section’

- Cross sections depend on
  - nature of projectile (electrons scatter off a proton more sharply than a neutrino)
  - nature of scattering interaction (elastic, inelastic ⇒ many different possibilities)
Cross Sections

- **Elastic scattering:**
  - Kinetic energy of scattering system remains unchanged; only directions of scatters are changed in the interaction (with a particle, with a potential)
- **Inelastic scattering:** kinetic energy is not conserved (lost or gained in the collision)
- **In particle collisions:** typically, several final states
  - \[ e + p \rightarrow e + p + \gamma \; ; \; \rightarrow e + p + \pi^0 \; ; \; \ldots \]
  - each channel has its ‘exclusive’ scattering cross section
  - total (‘inclusive’) cross section \( \sigma_{\text{tot}} = \sum \sigma_i \); 
- **Cross section depends in general on velocity of projectile**
  - naively: \( \sigma \) proportional to time in vicinity of target \( \Rightarrow \sigma \sim 1/v \)
  - strong modification, if incident particle has energy to form a ‘resonance’ (or an excitation): a short-lived, quasi-bound state
    - Frequently used procedure to discover short-lived particles
Hard-Sphere Scattering

• Particle scatters elastically on sphere with radius R

\[ b = R \sin \alpha \quad 2 \alpha + \theta = \pi \]
\[ \sin \alpha = \sin(\pi/2 - \theta/2) = \cos(\theta/2) \]
\[ b = R \cos(\theta/2) \quad \theta = 2 \cos^{-1}(b/R) \]
for \( b = 0 \rightarrow \theta = \pi \)

• If particle impinges with impact parameter between \( b \) and \( b + db \) ⇒ will emerge under angle \( \theta \) and \( \theta + d\theta \)
Scattering into Solid Angle $d\Omega$

- More generally, if particle passes through an infinitesimal area $d\sigma$ at impact parameter $b$, $b+db \Rightarrow$ will scatter into corresponding solid angle $d\Omega$

- Differential cross section $D(\theta)$:
  - differential with respect a certain parameter: solid angle; momentum of secondary particle…

\[
d\sigma = D(\theta) \, d\Omega
\]
\[
d\sigma = \left| b \, db \, d\phi \right|, \quad d\Omega = \left| \sin \theta \, d\theta \, d\phi \right|
\]
\[
D(\theta) = \frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin \theta} \left( \frac{db}{d\theta} \right) \right|
\]
(areas, solid angles are intrinsically positive, hence absolute value signs)
Cross Sections

- Suppose particle (e.g. electron) scatters off a potential (e.g. Coulomb potential of stationary proton)

- Scattering angle $\theta$ is function of impact parameter $b$, i.e. the distance by which the incoming particle would have missed the scattering center
Scattering: Some examples

• Hard-sphere scattering

\[
\frac{db}{d\theta} = -\frac{R}{2} \sin\left(\frac{\theta}{2}\right)
\]

\[
D(\theta) = \frac{R b \sin(\theta/2)}{2 \sin \theta} = \frac{R^2}{2} \frac{\cos(\theta/2) \sin(\theta/2)}{\sin \theta} = \frac{R^2}{4}
\]

• Total cross section \( \sigma = \int d\sigma = \int D(\theta) \, d\Omega \)

- \( \sigma_{\text{total (hard-sphere)}} = \int \frac{R^2}{4} \, d\Omega = \pi R^2 \)
  - for hard-sphere: \( \pi R^2 \) obviously presents the scattering area

• Concept of ‘cross section’ applies also to ‘soft’ targets
Rutherford Scattering

- Particle with charge $q_1$ and kinetic energy $E$ scatter off a stationary particle with charge $q_2$

- Classically: 
  \[ b = \frac{q_1 q_2}{2E} \cot\left(\frac{\theta}{2}\right) \quad b_{\text{min}} = \frac{q_1 q_2}{2E} \]

- Differential cross section: 
  \[ D(\theta) = \left(\frac{q_1 q_2}{4E \sin^2(\theta/2)}\right)^2 \]

- Total cross section: 
  \[ \sigma = 2\pi \left(\frac{q_1 q_2}{4E}\right)^2 \int_0^\pi \frac{1}{\sin^2(\theta/2)} \sin \theta \, d\theta = \infty \]

  infinite, because Coulomb potential has infinite range

- Remember: in calculation of energy loss of a charged particle one imposes a cut off ($b_{\text{max}}$), beyond which no atomic excitations are energetically possible.
Cowan, Reines Experiment of Neutrino Discovery: estimate of event rate

neutrino flux: $5 \times 10^{13}$ neutrinos/ cm$^2$ s
Tanks was located 11 m from reactor core, 12 m blow surface (why?)
v absorbed in water (200 l); effective area was 3000 cm$^2$; depth ~ 70 cm
Cross section for absorption (inverse $\beta$-decay $\sigma = 6 \times 10^{-44}$ cm$^2$

Energy threshold for reaction: 1.8 MeV
3% of the reactor neutrinos are above threshold
Cowan, Reines Experiment of Neutrino Discovery: estimate of event rate

\[ N_a = 6.02 \times 10^{23} / \text{mol} \ldots 18 \text{ g H}_2\text{O} \]

Number of scatterers \( N_S \)/ cm\(^2\)

\[ N_S = N_A L \rho / 18 = 2.3 \times 10^{24} \]

Probability for interaction =

Effective ‘cross section’/ cm\(^2\) =

\[ P = N_S \sigma = 2.3 \times 10^{24} \times 6 \times 10^{-44} \]

\[ = 1.4 \times 10^{-19} \]

Rate = \( P \times \text{Flux} \times \text{Detector Area} \times \text{Threshold} = 6.3 \times 10^{-4} \)

Rate is \(~ 2 \) events/ hour…was actually observed!
Luminosity

- Luminosity $\mathcal{L}$: number of particles per unit time and unit area:
  
  $$[\mathcal{L}] = \text{cm}^{-2} \text{s}^{-1}$$

- $dN = \mathcal{L} \, d\sigma$ .... number of particles per unit time passing through area $d\sigma$ and $b$, $b+db$, which is also number per unit time scattered into solid angle $d\Omega$: $dN = \mathcal{L} \, d\sigma = \mathcal{L} \, D(\theta) \, d\Omega$

- Setting up a scattering experiment with a detector covering a solid angle $d\Omega$ and counting the number of particles per unit time $dN$

- Measurement of differential cross section $\frac{d\sigma}{d\Omega} = \frac{dN}{\mathcal{L} \, d\Omega}$

- If $d\Omega \rightarrow 1$ \quad $N = \sigma \mathcal{L}$

- At LHC: $\mathcal{L} = 10^{34} \text{ cm}^{-2} \text{s}^{-1}$, $\sigma = 100 \text{ mb} = 10^{-25} \text{ cm}^2$ \quad $N = 10^9$ coll/s
Luminosity vs. Collision Rate

**Collision rate:**

- Probability $P$ for interaction = Effective ‘cross section’/ cm$^2$ = $P = N_S (\text{Number of scatters/ cm}^2) \sigma$

  Collision Rate Rate = $P \times \text{Flux} \times \text{Detector Area}$

**Luminosity $L$**: dimension cm$^{-2}$ sec$^{-1}$

- Groups these parameters into one Parameter, Luminosity
- In colliders: includes the details about the shape and particle density of the colliding beams
- Defined as the proportionality between the rate $R_i$ of process $i$ with
  $R_i = L \sigma_i$

**Time-Integrated Luminosity:**

- In 2011: LHC ‘delivered’ 5 fb$^{-1}$
- 1 fb$^{-1} = 10^{24} \times 10^{15}$ cm$^{-2}$
- Total number of collisions: $N = L \sigma_i = 5 \times 10^{39} \times 10^{-25} = 5 \times 10^{14}$
Calculating Decay Rates and Cross Sections

• **Calculation needs two ingredients**
  - the amplitude $\mathcal{M}$ for the process (dynamics)
  - the phase space available (kinematics)

• $\mathcal{M}$ depends on the physics, type of interaction, reaction, …

• **Phase space: ‘room to maneuver’**
  - heavy particle decaying into light particles involves a large phase space factor
  - neutron decaying into $p + e^- + \bar{\nu}_e$ has a very small phase space factor

• **Fermi ‘Golden Rule’**
  - transition rate = $(\text{phase space}) \cdot |\text{amplitude}|^2$
    - Non-relativistic version formulated with perturbation theory
    - can be derived with relativistic quantum field theory
Golden Rule for Decays

- Particle 1 (at rest) decaying into 2, 3, 4, …. n particles
  \[ 1 \rightarrow 2 + 3 + 4 + \ldots + n \]

- Decay rate \( \Gamma \)

\[
\Gamma = \frac{S}{2\hbar m_j} \int |M|^2 (2\pi)^4 \delta^4 (p_1 - p_2 - p_3 \ldots - p_n) \times \prod_{j=2}^{n} 2\pi \delta (p_j^2 - m_j^2 c^2) \theta (p_j^0) \left( \frac{d^4 p_j}{2\pi^4} \right)
\]

- \( m_j, p_j \) is mass, four-momentum of ith particle; \( S \) is statistical factor to account for identical particles in final state; (if no identical particles in final state \( S = 1 \))

- \( |M|^2 \) is amplitude squared; contains the dynamics

- The rest is ‘Phase Space’
Phase Space

- Integration over all outgoing four-momenta subject to three kinematical constraints
  - each outgoing particle lies on its mass shell: $p_j^2 = m_j^2 c^2$
  - each outgoing energy is positive: $p_j^0 = E_j / c > 0$; this is ensured by the $\theta$ – function ($\theta(x) = 0$ for $x<0$; $\theta(x) = 1$ for $x>0$)
  - energy and momentum is conserved $p_1 = p_2 + p_3 \ldots + p_n$; this is ensured by delta function $\delta^4(p_1 - p_2 - \ldots - p_n)$

- Practical points:
  - $d^4 p = dp^0 d^3 \vec{p}$
  - $\delta(p^2 - m^2 c^2) = \delta[(p^0)^2 - p^2 - m^2 c^2] \Rightarrow p^0$ integrals doable, using $\delta$ function
  - $\Gamma = \frac{s}{2\hbar m_i} \int |M|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \ldots - p_n) \prod_{j=2}^{n} \frac{1}{2\sqrt{p_j^2 + m_j^2 c^2}} \frac{d^3 \vec{p}_j}{(2\pi)^3}$
Two-Particle Decays

• For two particles in final state, previous expression reduces to

\[ \Gamma = \frac{S}{32 \pi^2 \hbar m_1} \int |M|^2 \frac{\delta^4(p_1 - p_2 - p_3)}{\sqrt{p_2^2 + m_2^2 c^2}\sqrt{p_3^2 + m_3^2 c^2}} d^3 \vec{p}_2 \times d^3 \vec{p}_3 \]

• After several pages of algebra and integration

\[ \Gamma = \frac{S|\vec{p}|}{8 \pi \hbar m_1^2 c} |M|^2 \]

• Surprisingly simple: phase space integration can be carried out without knowing functional form of \( M \) \( \Rightarrow \) two-body decays are kinematically determined: particles have to emerge back-to-back with opposite three-momenta

• no longer true for decays into more than two particles
Golden Rule for Scattering

- Particles 1 and 2 collide, producing 3, 4, ..., n particles
  \[1 + 2 \rightarrow 3 + 4 + \ldots n: \text{after performing } p_j^0 \text{ integrals}\]

\[\sigma = \frac{\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |M|^2 \left(2\pi\right)^4 \delta^4(p_1 + p_2 - p_3 - \ldots - p_n) \times \prod_{j=3}^{n} \frac{1}{2\sqrt{p_j^2 + m_j^2 c^2}} \left(2\pi\right)^3\]

same definitions, procedures as for decay

- Two-body scattering in CM frame 1+2 \(\rightarrow\) 3+4; \(\vec{p}_2 = -\vec{p}_1\)

\[\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S |M|^2 |\vec{p}_f|}{(E_1 + E_2)^2 |\vec{p}_i|} \quad \text{with } |\vec{p}_i|, |\vec{p}_f| \quad \text{the magnitude of either incoming or outgoing particle}\]

- Dimensions and units
  - decay rate \(\Gamma = 1/\tau [\text{sec}^{-1}]\);
  - cross section: area [cm²], 1 barn = 10⁻²⁴ cm²
  - \(ds/d\Omega \quad \ldots\) barns/steradian
  - \(\mathcal{M} \quad \ldots\) units depend on total number of particles involved \([\mathcal{M}] = (mc)^{4-n}\)